9.9 Improper Integral

1070. The definite integral $\int_a^b f(x)dx$ is called an improper integral if

- a or b is infinite,
- $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

1071. If $f(x)$ is a continuous function on $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{n\to\infty}\int_a^n f(x)dx.$$

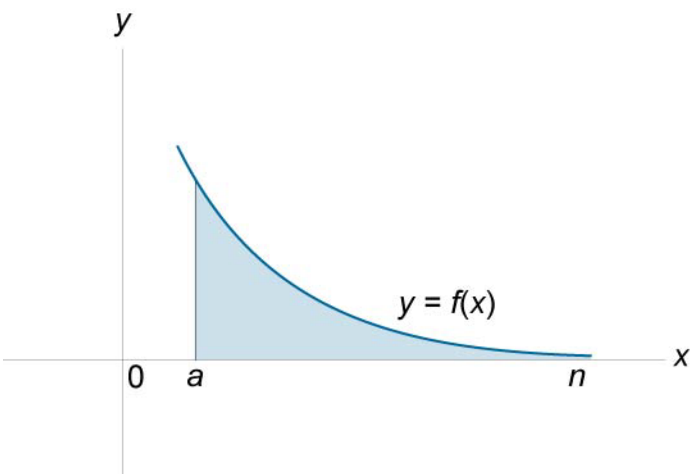

Figure 184.

1072. If $f(x)$ is a continuous function on $(-\infty,b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{n \to -\infty} \int_{n}^{b} f(x)dx .$$

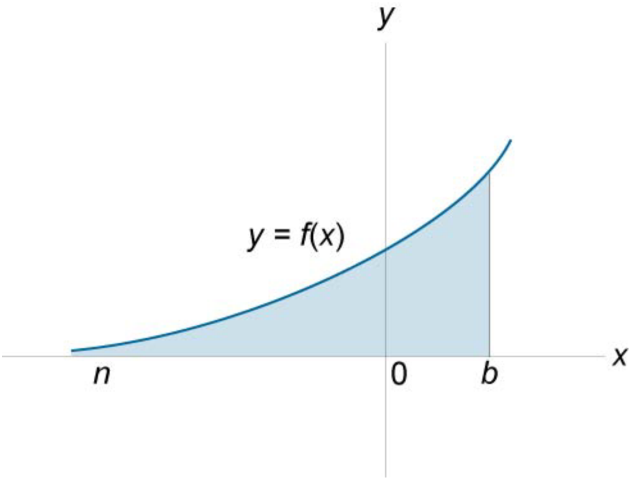

Figure 185.

Note : The improper integrals in 1071, 1072 are convergent if the limits exist and are finite; otherwise the integrals are divergent.

1073. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

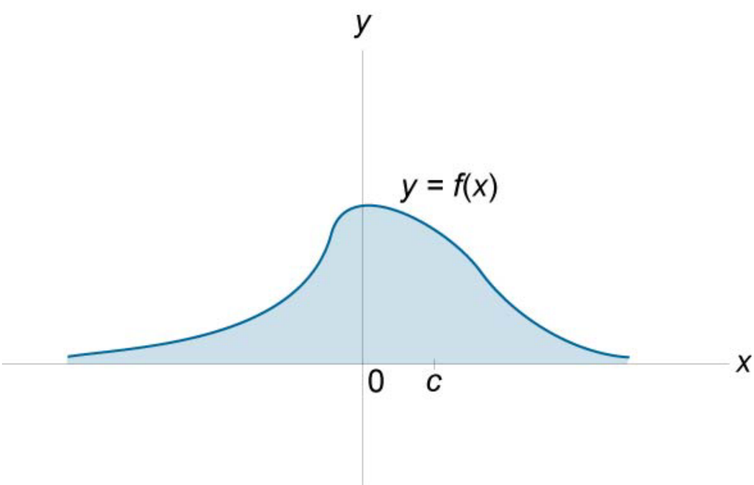

Figure 186.

If for some real number c, both of the integrals in the right side are convergent, then the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

1074. Comparison Theorems
Let $f(x)$ and $g(x)$ be continuous functions on the closed interval $[a,\infty)$. Suppose that $0 \le g(x) \le f(x)$ for all x in $[a,\infty)$.

- If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is also convergent,
- If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is also divergent.

1075. Absolute Convergence

If $\int_a^{\infty} |f(x)|dx$ is convergent, then the integral $\int_a^{\infty} f(x)dx$ is absolutely convergent.

1076. Discontinuous Integrand
Let $f(x)$ be a function which is continuous on the interval $[a,b)$ but is discontinuous at $x = b$. Then

$$\int_a^{b} f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx$$

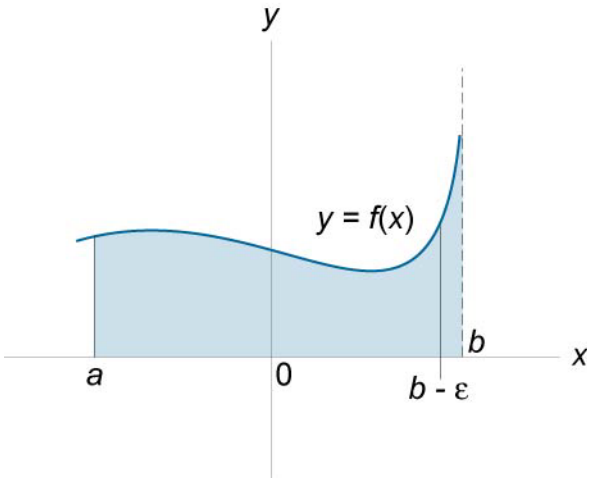

Figure 187.

1077. Let $f(x)$ be a continuous function for all real numbers x in the interval $[a,b]$ except for some point c in (a,b). Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^{b} f(x)dx.$$

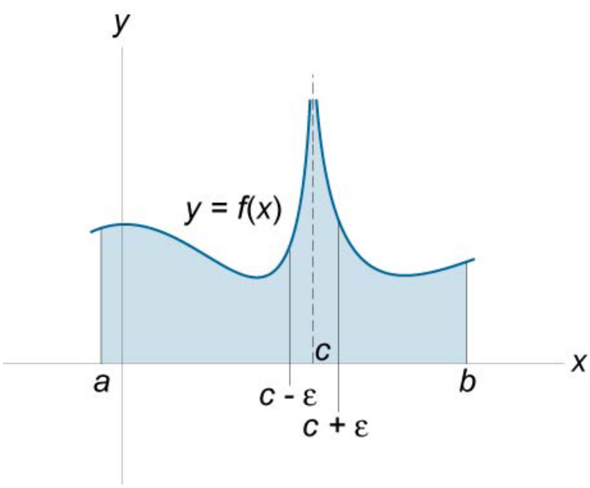

Figure 188.